

Algorithm xxx: jMarkov: an Integrated Framework for Markov Chain Modeling

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Markov chains (MC) are a powerful tool for modeling complex stochastic systems. Whereas a number of tools exist for *solving* different types of MC models, the first step in MC modeling is to define the model parameters. This step is however error prone and far from trivial when modeling complex systems. In this article we introduce jMarkov, a framework for MC modeling that provides the user with the ability to define MC models from the basic rules underlying the system dynamics. From these rules, jMarkov automatically obtains the MC parameters and solves the model to determine steady-state and transient performance measures. The jMarkov framework is composed of four modules: (i) the main module supports MC models with a finite state space; (ii) the jQBD module enables the modeling of Quasi-Birth-and-Death processes, a class of MCs with infinite state space; (iii) the jMDP module offers the capabilities to determine optimal decision rules based on Markov Decision Processes; and (iv) the jPhase module supports the manipulation and inclusion of phase-type variables to represent more general behaviors than that of the standard exponential distribution. In addition, jMarkov is highly extensible, allowing the users to introduce new modeling abstractions and solvers.

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1. INTRODUCTION

Stochastic modeling has become an important tool to analyze complex systems subject to uncertainty, which are prevalent in many areas, including manufacturing, logistics, computer and telecommunications networks, among others. A stochastic model can be used to perform what-if analyses to understand the effect of critical design parameters on overall system metrics, such as performance or reliability. Within the set of stochastic models, Markov chains (MC) have received particular attention due to their modeling flexibility, analytical tractability, and amenability for numerical analysis.

For these reasons, a number of software tools have been proposed for building and solving MC models. In fact, most of these tools focus on providing numerical methods to *solve* the model to obtain the MC stationary measure, from which relevant steady-state metrics can be computed. However, before solving the model, one first needs to determine the model parameters, such as the MC generator matrix, which must be provided as input to the solvers, but this task is far from trivial when modeling complex systems. In this paper, we introduce jMarkov, an extensible object-oriented tool for the modeling and solution of MC models. The key distinguishing feature of jMarkov lies in its *modeling*

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capabilities, where the user needs only to define the rules that govern the system dynamics. Based on these rules, jMarkov *builds the model parameters*, solves the model, and computes the measures of performance.

Furthermore, jMarkov not only supports finite MCs, but it is unique as a modeling tool for Quasi-Birth-and-Death processes (QBD) and Markov Decision Processes (MDP). QBDs allow the user to consider models with an infinite state space, whereas MDPs support the computation of optimal decision policies. While a number of tools provide *solvers* for these models, jMarkov takes on the *modeling* step, for which no alternative tools exist. Thus, the user can offload to jMarkov the cumbersome and error-prone procedure of building the model parameters. In addition, the framework is complemented by a module to manipulate phase-type (PH) distributions, a class of distributions that can be incorporated in MC models to consider behaviors more general than those allowed by the standard exponential distribution. An early version of some of these modules was described in [Riaño and Góez 2006; Pérez and Riaño 2006].

A key feature in the design of jMarkov is the separation of modeling from solving, allowing the analyst to concentrate on describing the system without needing to worry about the implementation of the solution algorithms. In this regard, the jMarkov framework resembles software packages for mathematical programming that offer Java libraries, like Gurobi [Gurobi Optimizaton 2014] and CPLEX [CPLEX Optimizer 2015], which the user employs to specify an optimization problem, without needing to know the details of the underlying solution algorithms. Yet, the platform is flexible enough to allow new solvers to be implemented and added by third-party developers.

We developed the jMarkov framework in Java [Sun Microsystems 2006], and designed it to make extensive use of the abstraction principle that Java is built on. Java makes a great choice to enable the extensibility of jMarkov, thanks to its object-oriented nature. In addition, Java is a widely-used, platform independent, and well-supported language, allowing a large audience to use jMarkov without the need to learn a new programming language. Also, the extensibility of jMarkov is further supported by its open-source release and the availability of many examples (currently over 50) and detailed documentation. jMarkov is publicly available as a project of the Computational Infrastructure for Operations Research (COIN-OR). Source code, documentation, examples, and case studies are available at [jMarkov website 2016].

The following sections present the four main modules, jMarkov, jQBD, jMDP, and jPhase, followed by numerical experiments and a description of extensions and applications.

2. THE JMARKOV MODULE

In this section we describe the main features of the jMarkov core module, which focuses on finite MC models. After a short review of MCs, we describe in detail the state-space search algorithm that enables the automatic model-building feature of jMarkov. We explain how jMarkov allows a user to exploit this algorithm by implementing a few classes and methods to describe the system state and the events that modify the state. To this end, we also illustrate the main classes in jMarkov that a user will typically rely on to build MC models. This description also serves as a stepping stone for the extensions introduced in sections 3 and 4 to handle QBD and MDP models.

2.1. Background

A homogeneous Continuous Time Markov Chain (CTMC) [Kulkarni 1995] is a stochastic process $\{X(t), t \geq 0\}$, defined on a countable state space S , that satisfies the Markov property. The jMarkov module focuses on irreducible CTMCs with finite S , and the following discussion assumes this condition. A CTMC with $n = |S|$ states is completely determined by two parameters: i) its initial distribution (row) vector α , with entries $\alpha_i = \mathbb{P}\{X(0) = i\}$ for $i \in S$, i.e., the probability that the system is in state i at time 0; and ii) its generator matrix $Q \in \mathbb{R}^{n \times n}$, with non-positive diagonal components such that $-q_{ii}$ is the exponential holding rate for state i , whereas the off-diagonal elements q_{ij} represent the transition rate from state i to state j . Since $q_{ii} = \sum_{j=1, j \neq i}^n q_{ij}$, one can equivalently define the MC by relying on the rate matrix T , which has the same entries as Q , except for its diagonal elements, which are zero. In fact, the objective of the state-space search algorithm implemented by jMarkov is to determine the matrix T .

From α and Q , it is possible to describe the transient behavior of the CTMC, which is given by the matrix $P(t) \in \mathbb{R}^{n \times n}$ with entries $p_{ij}(t) = \mathbb{P}\{X(t+s) = j | X(s) = i\}$, $i, j \in S$, i.e., the probability that the system is in state j at time t given that it is started in state i at time 0. This matrix can be computed as $P(t) = e^{Qt}$ for $t \geq 0$. In addition, it is also of interest to determine the stationary distribution of the process, i.e., the long-term fraction of time that the process spends in state i , which we label π_i .

The stationary distribution $\pi = [\pi_1, \pi_2, \dots]$ of an irreducible CTMC is obtained as the solution to the system of equations $\pi Q = \mathbf{0}$, $\pi \mathbf{1} = 1$, where $\mathbf{1} \in \mathbb{R}^{n \times 1}$ is a column vector of ones.

2.2. Algorithms

2.2.1. Building Large-scale Markov Chains. Transitions in a CTMC are triggered by the occurrence of events, for instance arrivals and departures in a queueing model. The matrix T can be decomposed as $T = \sum_{e \in \mathcal{E}} T^{(e)}$, where $T^{(e)}$ contains the transition rates associated with event e , and \mathcal{E} is the set of all possible events that may occur. In large systems, it is not easy to know in advance all the reachable states in the model and the transitions among them. However, it is possible to determine what events occur in each state, and the destination states produced by each transition when it occurs, thus obtaining the $T^{(e)}$ matrices that compose T . jMarkov works based on this observation, using the *BuildRS* algorithm [Ciardo 2000], which builds the state space and the rate matrix T by a deep exploration of the state graph. Algorithm 1 illustrates the *BuildRS* algorithm, which starts from a single state i_0 and the set of events \mathcal{E} to determine the full state space S and the rate matrix T . The algorithm goes through each state not yet analyzed (set \mathcal{U}), and determines whether each event $e \in \mathcal{E}$ is active or not in that state, according to the active method. If an event is active, the next step is to determine the set \mathcal{D} of all reachable states when this event occurs, by relying on the *dests* method. For each state in \mathcal{D} the algorithm then uses the method *rate* to determine the associated transition rate, and adds all newly found states to the set \mathcal{U} for further analysis. The algorithm stops when no states remain to be analyzed in \mathcal{U} . This algorithm therefore requires the definition of states, events, and of the methods *active*, *dests*, and *rate*. These are precisely the definitions that jMarkov requires the user to code as Java classes and methods. Once these basic methods are defined, jMarkov builds the matrix T , from which it derives the matrix Q and computes the stationary distribution and the associated measures of performance. Notice that the *BuildRS* algorithm underlying jMarkov slightly differs from the one in [Ciardo 2000] in that each event can trigger transitions to multiple states instead of a single one, as captured by the set \mathcal{D} returned by the method *dests*.

2.2.2. Measures of Performance. When studying MCs, the analyst is usually interested in the transient and steady-state behavior of a set of measures of performance (MOPs). This is accomplished by attaching rewards to the model. Let r be a column vector such that $r(i)$ represents the expected rate at which the system receives rewards whenever it is in state $i \in S$. For example, in queueing systems $r(i)$ might represent the number of entities in the system, or the number of busy servers, when the state is i . The expected reward rate at time t is thus $E(r(X(t))) = \alpha P(t)r$. Similarly, for an irreducible CTMC, the long run rate at which the system receives rewards is given by

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{E}(r(X(s))) ds = \pi r.$$

jMarkov provides mechanisms to define this type of rewards and can compute both transient and steady-state MOPs. There are other types of rewards, like the expected time in the system, which are computed using Little's law [Little 1961]. As these results rely on the CTMC being irreducible, jMarkov throws an exception of the type `NotUniChainException` in case the resulting CTMC is reducible.

2.3. Architecture

To better describe the capabilities of jMarkov, this section provides some details of its architecture, which consists of three main packages: *Build*, *Basic*, and *Solvers*.

2.3.1. The Build Package. The *Build* package plays a central role in jMarkov since it contains the classes that take care of building the state space and transition matrices. As depicted in Table I, the main classes are `MarkovProcess`, `SimpleMarkovProcess`, and `GeomProcess`. The class `MarkovProcess` implements the *BuildRS* algorithm described in the previous section, while `SimpleMarkovProcess` extends this class and defines the abstract methods *active*, *dests*, and *rate*. A typical user will therefore extend the `SimpleMarkovProcess` class and implement these three abstract methods. Alternatively, the user could directly extend the `MarkovProcess` class, which requires the implementation of the *activeTransitions* method. This method combines the *dests* and *rate* methods mentioned before, determining at once, for each state and event, the set of destination states and the associated rates. This implementation thus avoids traversing the set \mathcal{D} in Algorithm 1 twice, first to determine the destination states and then to determine the transition rates. While more efficient, this implementation may also require some previous experience with the jMarkov modeling approach, and we therefore expect users to extend the `SimpleMarkovProcess` class, implementing the *active*, *dests*, and *rate* methods separately. Both `SimpleMarkovProcess` and `MarkovProcess` contain tools to communicate with the

ALGORITHM 1: BuildRS algorithm

```

Input:  $\mathcal{U} = \{i_0\}, \mathcal{E}$  /* Input: the initial state and the set of events */
Output:  $S, T$  /* Outputs: the state space and the entries of  $T$  */
 $S = \emptyset$ ;
while  $\mathcal{U} \neq \emptyset$  do
  Choose an  $i \in \mathcal{U}$ ;
   $S := S \cup \{i\}$ ;
   $\mathcal{U} := \mathcal{U} \setminus \{i\}$ ;
  foreach  $e \in \mathcal{E}$  do
    if active( $i, e$ ) then /* Determine if the event  $e$  may occur while in state  $i$  */
       $\mathcal{D} := \text{dests}(i, e)$ ;
      foreach  $j \in \mathcal{D}$  do /* Check all reachable states found by dests */
         $T_{ij} := T_{ij} + \text{rate}(i, j, e)$ ;
        if  $j \notin S \cup \mathcal{U}$  then /* If  $j$  is unexplored add it to  $\mathcal{U}$  */
           $\mathcal{U} := \mathcal{U} \cup \{j\}$ ;
        end
      end
    end
  end
end

```

Table I. jMarkov Main Packages and Classes

| Build Package | Basic Package | Solvers Package |
|---------------------|-----------------|-------------------|
| MarkovProcess | Event | SteadyStateSolver |
| SimpleMarkovProcess | State | TransientSolver |
| GeomMarkovProcess | PropertiesEvent | GeomSolver |
| | PropertiesState | JamaSolver |
| | GeomState | Mtjsolver |
| | GeomRelState | MtjLogRedsolver |

solvers to compute steady state and transient solutions, and output them in different formats. Finally, the `GeomProcess` class extends the `MarkovProcess` class to model QBD processes, which we describe in Section 3.

2.3.2. The Basic Package. In addition to the `active`, `dests`, and `rate` methods, the other key step in setting up an MC model in jMarkov is the definition of events and states. To this end, the Basic package contains the abstract classes `State` and `Event`, which the user can extend to code the description of the states and events, respectively. The user has the freedom to choose the coding that best describes the states in a model, like any combination of integers, strings, etc. The only requirement here is to establish a complete ordering among the elements since, for efficiency, jMarkov works with ordered sets. For simplicity, however, this package also provides the `PropertiesState` class to describe the state as an array of integers, which is appropriate for many applications. Similarly, the `PropertiesEvent` class provides a simple event description in terms of integers. The two other classes of this package, `GeomState` and `GeomRelState`, are described in Section 3 as part of the jQBD module.

2.3.3. The Solvers Package. As we mentioned in the introduction, jMarkov separates modeling from solving. Once the model is built using the classes in the Build and Basic packages, it is solved with one of the solvers in the Solver package. As depicted in Table I, there are three (abstract) solvers, namely, `SteadyStateSolver`, `TransientSolver`, and `GeomSolver`. As the names indicate, the first two compute steady state and transient probabilities for finite MCs, whereas the latter is used for QBDs. The remaining classes listed in Table I extend either of these abstract classes to provide ready-to-use solvers. These solvers rely on the Java packages JAMA [Hicklin et al. 2005] and MTJ [Heimsund 2005], thus exploiting their capabilities to handle matrix operations for dense and sparse matrices, as well as a number of methods to solve the linear systems underlying the solution of the MC model. Notice that the user can also implement other solvers, e.g., making use of other libraries or developing a solver for a specific class of MC models. Alternatively, it is possible to export the obtained model and employ solvers available in other tools. This can be achieved for instance by using standard matrix exchange formats, or by directly calling the jMarkov libraries from other environments such as MATLAB. These features, not present in other MC modeling packages, highlight the flexibility of jMarkov.

2.3.4. *The Graphical User Interface.* Another feature of jMarkov is its graphical user interface (GUI). This GUI presents the measures of performance, displays debugging messages, provides a full list of the states found in the state generation process, as well as a spreadsheet-like view of the rate matrix T . This information can be very valuable to the user when developing a new model, as it allows for a detailed view of the model parameters. For instance, the user can verify, for a small-scale example, that all the transition rates are as expected, or that the transitions obtained for a subset of states, and their associated rates, comply with the high-level definition. The GUI thus supports the user in the verification of the model correctness.

2.4. Related Work

There are other academic software packages similar to jMarkov. MARCA [Stewart 1996] is a Fortran 77 package that supports modeling large MCs and provides a large number of numerical solution procedures. One limitation of MARCA is that, to make use of its model-building capabilities, it requires the user to define the state as a set of integers and to provide a description of the transitions in terms of buckets. This implies determining, for each entry in the state descriptor (bucket), the destination buckets when a transition occurs. While this description may be simple to obtain for certain models, it may be difficult or unfeasible to provide such description for a complex model. Another package is SMART [Ciardo et al. 2002], which provides a modeling language with its own syntax for the definition of the MC. This requires the user to learn a new language to make use of SMART. SMART also offers built-in solvers and supports both continuous and discrete time MCs. Another issue, shared by both MARCA and SMART, is that they are no longer supported, making the tools and their documentation hard to access. Another tool is DNAmaca [Knottenbelt 1996], which also defines a modeling language and a parser that writes the corresponding C++ code and compiles the classes to obtain the state space generator for the model. This generator feeds the tools to compute the steady state probabilities and measures of performance. The DNAmaca language is actually oriented towards Petri net analysis, and DNAmaca is now embedded within the Java Platform Independent Petri net Editor (PIPE) [Dingle et al. 2009]. As SMART, DNAmaca requires the user to use a specific language to describe the model. Also, it does not provide transient analysis. In addition to the above, we must point out that all these tools offer a rigid combination of modeling and solution, without easy means to add new solvers or alternative state-space search method. In contrast, jMarkov is an easily extensible tool, with an architecture that readily allows for novel solvers or other state-space search methods. Also, jMarkov is an open-source project, with easy to access source code and documentation, and it is implemented in Java, a widely used programming language. Moreover, the state definition in jMarkov is fairly general, allowing for descriptors closer to the actual system under analysis. Finally, as we shall see in the next sections, jMarkov is not limited to modeling finite MCs. It also supports QBD and MDP models, which are not currently supported by any other tool.

There are also commercial software packages that include MC modules like RAM Commander [Advanced Logistic Department 2013] and ITEM Toolkit, [ITEM Software (USA) Inc. 2011], which provide graphical interfaces to build MCs and find the steady state probabilities. However, these tools are designed to graphically build models with a few states and not to build and solve large MCs. These graphical interfaces work well to manually build small models, but they do not provide support to automatically build large-scale models.

3. THE JQBD MODULE

While finite MC models offer great flexibility, in many applications it is natural to assume that some variables can take a countably infinite number of values. This is the case for instance in queueing applications where the queue-length is assumed to be countably infinite. To model such systems, jMarkov offers support, via the jQBD module, for Quasi-Birth-and-Death (QBD) processes [Latouche and Ramaswami 1999], which are MCs with an infinite state space, but with a very specific structure that makes them numerically tractable. Notice that, due to their infinite state space, the *BuildRS* algorithm introduced in Section 2 cannot be directly applied as it would never terminate. In this section we introduce QBDs, and explain how the *BuildRS* algorithm can be modified to enable jMarkov to model them.

3.1. Background

Consider a MC $\{X(t) : t \geq 0\}$ with a two-dimensional state space $\mathcal{S} = \{(n, i) : n \geq 0, 0 \leq i \leq m\}$. The first coordinate n is called the *level* of the process, and the second coordinate i is called the *phase*. We assume that the number of phases m is finite. In queueing applications, the level usually represents the number of items in the system, whereas the phase may represent the stages of a service process. While

in a birth-and-death process the only allowed transitions are to the two adjacent states [Kulkarni 1995], in a QBD a transition is allowed to any state in the same or in adjacent *levels*. Also, for $n \geq 1$ the transition rates are independent of the level n . Therefore, the generator matrix Q has the following structure

$$Q = \begin{bmatrix} B_{00} & B_{01} & & & \\ B_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \quad (1)$$

where, as usual, the rows add up to 0. In general, the level zero might have a number of phases $m_0 \neq m$. Note that the matrix B_{00} has size $m_0 \times m_0$, whereas B_{01} and B_{10} are matrices of sizes $(m_0 \times m)$ and $(m \times m_0)$, respectively. Assuming the QBD is ergodic, there exists a steady state distribution π that is the unique solution $\pi Q = 0$, $\pi \mathbf{1} = 1$. Partitioning the vector π by levels $\pi = [\pi_0, \pi_1, \dots]$, these sub-vectors satisfy $\pi_{n+1} = \pi_n \mathbf{R}$, $n > 1$. Here \mathbf{R} is a square matrix of order m , called the rate matrix [Neuts 1981], that is the minimal non-negative solution to the equation $A_0 + \mathbf{R}A_1 + \mathbf{R}^2A_2 = 0$.

3.2. Algorithms

3.2.1. State-space Search for QBD processes. As with finite MCs, QBDs can be modeled with the jQBD module by defining states and events, and describing the system dynamics via the *active*, *dests*, and *rate* methods. However, due to the infinite state space of the QBD, we cannot rely on the *BuildRS* algorithm directly. To handle QBDs we therefore extend the *BuildRS* algorithm via two key modifications. First, we modify the *dests* method to incorporate the concept of level that defines a QBD. Since from a state (n, i) , transitions are allowed only to states in levels $n-1$, n , and $n+1$, the *dests* method determines the destination state as a *relative* state $(-1, j)$, $(0, j)$, $(1, j)$. This change requires replacing the *State* class for finite MCs by two classes: *GeomState* for absolute states that belong to a certain level $n \geq 0$; and *GeomRelState* for relative states that indicate whether in the destination state the level increases, decreases or remains the same. Second, rather than searching the whole state space, we limit the search to the first three levels of the generator matrix (1), from which we can extract the matrices B_{00} , B_{01} , B_{10} , A_0 , A_1 , and A_2 . These modifications are used by the *GeomProcess* class to override the *BuildRS* algorithm and obtain the parameters of the QBD, which it passes to the solver to compute the matrix \mathbf{R} , as described in the next section.

3.2.2. Computing the rate matrix. A number of iterative algorithms have been proposed to compute the matrix \mathbf{R} . jQBD provides an implementation of the logarithmic-reduction algorithm [Latouche and Ramaswami 1999] in the class *MtjLogRedSolver*. We choose this algorithm as it is the only one, together with cyclic-reduction [Bini et al. 2005], to offer a quadratic convergence rate, a feature particularly beneficial for models that are close to null-recurrent but stable, as in queues with a very high utilization.

Once the rate matrix \mathbf{R} has been found, the π_0 and π_1 vectors are determined by solving the following linear system of equations

$$[\pi_0 \ \pi_1] \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & A_1 + \mathbf{R}A_2 \end{bmatrix} = [0 \ 0], \quad \pi_0 \mathbf{1} + \pi_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{1} = 1. \quad (2)$$

3.2.3. Measures of performance for QBDs. We consider two types of MOPs that can be defined in a QBD model. The first type can be seen as a reward r_i received whenever the system is in phase i , independent of the level $n \geq 1$. The long-run value for such a measure of performance is computed as $\sum_{n=1}^{\infty} \pi_n \mathbf{r} = \pi_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{r}$, where \mathbf{r} is a size- m column vector with components r_i . The second type of reward has the form nr_i whenever the system is in phase i of level n . Its long-run value is $\sum_{n=1}^{\infty} n \pi_n \mathbf{r} = \pi_1 \mathbf{R} (\mathbf{I} - \mathbf{R})^{-2} \mathbf{r}$. The first type of performance measure can be used, for instance, to determine the steady-state utilization of a server whose state is described through the phase variable. A common example of the second type of performance measure is the mean queue length, where the queue length is modeled as the level variable. We conclude this section by highlighting that, thanks to its close integration with jMarkov, models built with jQBD can also exploit the GUI described in Section 2.3.4 to debug and verify the correctness of the model.

3.3. Related Work

Whereas a number of tools exist to solve QBDs, the modeling capabilities of jMarkov are not available in any other tool. Probably the most well-known tool to analyze QBDs is SMCSolver [Bini et al. 2009], a software with versions in both Fortran 90 and Matlab, where the former also offers a graphi-

cal user interface. It contains state-of-the-art algorithms for solving QBDs, as well as other structured MCs (GI/M/1 and M/G/1-types). Similarly, MAMSolver [Riska and Smirni 2007] offers C++ and Matlab implementations of methods to solve QBD, M/G/1 and G/M/1-type Markov chains. In particular, it implements the ETAQA method [Riska and Smirni 2007] to efficiently compute average metrics such as the mean queue length when modeling queueing systems. Also, Butools [Butools 2016] offers methods to analyze QBDs and M/G/1-type Markov chains, as well as Markovian fluid models, implemented in Matlab and Python. While jMarkov offers support for QBDs only, its distinctive feature lies in its modeling capabilities, which none of the other tools have. In fact, existing tools focus on *solving* the model, assuming as input the matrices A_0 , A_1 , and A_0 , and the other submatrices that define the generator Q in (1). jMarkov is the only one with *modeling* capabilities, where these matrices are built from first principles.

4. THE JMDP MODULE

Markov Decision Processes (MDP) [Puterman 1994] enable the modeling and solution of complex multi-stage decision problems in the presence of uncertainty. MDPs build on the theory of Markov chains to model the random system dynamics, and incorporate a decision-maker that takes actions at each stage, affecting both the system evolution and the associated rewards or penalties. The jMarkov framework includes the jMDP module for modeling and solving MDPs. While there are several tools available for *solving* discrete-time MDPs, jMDP is unique in its capabilities for *modeling* MDPs. This includes support for modeling problems with or without events and for modeling both discrete-time and continuous-time problems. The next sections provide a short introduction to MDPs and describe in detail how the jMDP module extends the state-space search methods introduced in Section 2 to automatically build MDP models. We also describe the solution methods provided by jMDP to analyze the different types of MDP models supported. Examples of jMDP models are provided in [Riaño et al. b, Section 4].

4.1. Background

The problems that can be modeled and solved with the jMDP module can be classified in two categories based on the planning horizon: finite and infinite horizon problems. Infinite horizon problems can be in either discrete or continuous time. For each model type in these categories, jMDP provides a Java class for its modeling. Additionally, we provide classes to model MDP problems with events, and actions that depend on events, a technique that results in a more natural representation. In this section we present detailed definitions for discrete time, infinite horizon problems, whereas other problem types are described more briefly.

4.1.1. Infinite Horizon Discrete Time MDP Problems (DTMDP). Consider a discrete-space, discrete-time, bivariate random process $\{(X_n, A_n), n = 0, 1, \dots\}$. $X_n \in \mathcal{S}$ represents the state of the system at period n , and $A_n \in \mathcal{A}$ is the action taken at that period. The sets \mathcal{S} and \mathcal{A} are the spaces of all states and actions respectively, and we assume that both are finite. We also assume that all system parameters are stationary. The system state evolves according to the actions taken at each period, which leads the system to another state according to a probability distribution. We define $p_{ij}(a)$ as the probability of transitioning to state j whenever action a is taken and the system is in state i . A *deterministic policy* is a function d that assigns an action from the set \mathcal{A} , for each state. For each action a taken from state i , a finite cost $c(i, a)$ is incurred. Consequently it is possible to define a total expected cost $v^d(i)$ incurred when following policy d , called the *value function*, as

$$v^d(i) = \mathbb{E}_d \left[\sum_{n=0}^{\infty} c(X_n, A_n) \middle| X_0 = i \right], \quad i \in \mathcal{S}, \quad (3)$$

where \mathbb{E}_d is the expectation operator over the probability distribution associated with policy d .

The objective is to find a policy $d^* \in D$, where D is the set of all deterministic policies, that minimizes the objective function shown above, $v^*(i) = \inf_{d \in D} v^d(i)$. One can consider more general policies (randomized or history-dependent). However, it has been shown that for a large class of finite-state, finite-action MDPs there exists a deterministic policy that is optimal. For this reason, and for tractability, we restrict jMDP to consider only deterministic policies. The total cost criterion in (3) is only useful for problems with costs that converge to zero fast enough. jMDP also supports a discounted cost criterion, for a given discount factor $0 < \alpha < 1$, where each term in (3) is multiplied by α^n ; or a long-run average cost where the objective is the per-period average of (3).

Let $\mathcal{A}(i)$ be the set of *feasible actions* that can be taken from state i , and $\mathcal{S}(i, a)$ the set of reachable states from state i when action a is taken. Then $\mathcal{A}(i)$, $\mathcal{S}(i, a)$, $c(i, a)$ and $p_{ij}(a)$, together with the cost

criterion (and α if necessary) completely characterize a DTMDP. Note that $c(i, a)$ only needs to be defined for each feasible action $a \in \mathcal{A}(i)$ and $p_{ij}(a)$ is only defined for each feasible action $a \in \mathcal{A}(i)$ and each reachable state $j \in \mathcal{S}(i, a)$.

4.1.2. Finite Horizon MDP Problems (FHMDP). Now consider a discrete-time problem that only goes on for a finite number of periods $N < \infty$, called the *horizon* of the problem. In this case, we allow the parameters to vary over time. Specifically, each period n may have different feasible actions, reachable states, transition probabilities and costs. Therefore, the policies considered are also allowed to vary over time, so an action is selected for each state-period combination. We only consider the total cost criterion for FHMDP.

4.1.3. Continuous Time Infinite Horizon MDP Problems (CTMDP). These are a continuous time analog of DTMDPs, where instead of periods and transition probabilities the system dynamics are determined by time-homogeneous transition rates $\lambda_{ij}(a) = \lim_{h \rightarrow 0} P\{X(t+h) = j | X(t) = i, A(t) = a\}$. The Markov property implies that the sojourn time in state i under action a is exponentially distributed. In this case, costs can be a lump sum for each time an action is taken, or can be continuously incurred at a rate dependent on state and action. We consider the total, discounted and average cost criteria for CTMDP.

4.1.4. Problems Modeled with Events. All the previous problem types can be modeled in a more intuitive way using events. The jMDP module includes classes to model each of these problem types with or without events. We label the event-based version of each model by adding the suffix *ev*, e.g. FHMDPEv. Modeling MDPs with events implies that the transitions can be conditioned on certain events occurring at each period [Becker et al. 2004; Mahadevan et al. 1997; Feinberg 2004].

For example, consider a DTMDP, and let e be an event that triggers a state transition. Then the costs, reachable states, and transition probabilities will depend not just on the state and action taken, but also on the potential events e that can occur at each state. The value function (3) can also be written in terms of the event-dependent parameters. It has been shown [Mahadevan et al. 1997] that this DTMDP with events is equivalent to the standard DTMDP model presented earlier. jMDP also supports models where actions depend on the events, though we omit the details here in the interest of space.

4.2. Features and Algorithms

The jMDP module provides two functionalities to the user. First, it allows the user to model MDPs of all the types described in the previous section, in an efficient and intuitive manner. That is, following the same steps one would usually follow when defining the mathematical model. Second, it provides a suite of methods to solve such MDPs. In this section we discuss the algorithms and features associated with each of these functionalities.

4.2.1. Modeling Features and Algorithms. When modeling FHMDP or DTMDP the analyst generally follows these steps: characterize the states of the system, the events that can occur at each state, and the actions considered in each state; compute the conditional probability distribution of the destination state once an action has been taken and a given event occurs; and determine the immediate cost for every state, event, and action. jMDP allows the user to model MDPs by following these steps.

Our framework defines a computational class or method for each of these mathematical elements, which the user has to extend or implement in order to represent the model. To achieve this we leverage jMarkov's integrated structure and use the Basic package described in Section 2.3.2, which includes the abstract classes `Action`, `PropertiesAction`, and `Actions` to describe the actions in an MDP, analogous to the classes `State`, `PropertiesState`, and `States` to describe the states. We should note that this integrated structure means that if any new features, such as a class for novel state representations are added to the jMarkov Basic package, they become immediately available in jMDP and the rest of the modules.

The classes specific for MDP modeling are provided in a separate modeling package. Here we follow a hierarchical class structure where attributes and methods that are common to various classes are included only in the classes at the top of the hierarchy. This structure simplifies the extensibility of the module as new classes can exploit the methods already implemented in the existing classes. At the top of the hierarchy we have the abstract class `MDP`, which defines the basic procedures for MDP models, such as the `setSolver` and `getSolver` methods to define and access the solver, respectively. Then as we go down the hierarchy we implement more specific methods, for example the abstract class `InfiniteMDP` extends `MDP` and provides a number of methods, including for example a method to define feasible actions (`feasibleActions`). Finally, at the bottom of the hierarchy we have classes specific for each problem type described in Section 4.1, which exploit the methods provided by all the classes above

ALGORITHM 2: MDPBuildRS algorithm

```

Input:  $\mathcal{U} = \{i_0\}$ , /* Input: the initial state */
Output:  $\mathcal{S}$  /* Outputs: the state space */
 $\mathcal{S} = \emptyset$ ;
while  $\mathcal{U} \neq \emptyset$  do
  Choose an  $i \in \mathcal{U}$ ;
   $\mathcal{S} := \mathcal{S} \cup \{i\}$ ;
   $\mathcal{U} := \mathcal{U} \setminus \{i\}$ ;
  foreach  $a \in \text{feasibleActions}(i)$  do
    foreach  $j \in \text{reachable}(i, a)$  do
      if  $j \notin \mathcal{S} \cup \mathcal{U}$  then /* If  $j$  is unexplored add it to  $\mathcal{U}$  */
         $\mathcal{U} := \mathcal{U} \cup \{j\}$ ;
      end
    end
  end
end
end

```

it in the hierarchy. For example, to model a DTMDP, described in Section 4.1.1, the class DTMDP, has methods to define reachable states $\mathcal{S}(i, a)$ (reachable), immediate costs $c(i, a)$ (immediateCost), and transition probabilities $p_{ij}(a)$ (prob), but also uses methods from the superclass InfiniteMDP, such as feasibleActions to define the feasible actions $\mathcal{A}(i)$.

In order to build an DTMDP or FTMDP model in jMDP the user must extend the corresponding model class, as well as the State and Action classes to define the corresponding states and actions. Alternatively, users can exploit the existing PropertiesState or PropertiesAction classes, which define the state and action, respectively, as arrays of integers. Finally, the user has to implement the feasibleActions, reachable, prob and immediateCost methods.

With the definitions above, jMDP generates the corresponding MDP model. To achieve this, we extend the state-space exploration algorithm (*BuildRS*) described in Section 2.2.1, as depicted in Algorithm 2. Namely, instead of considering the possible events in each state, we incorporate the feasible actions and the reachable states defined in the MDP as the mechanisms to explore the state space. As we consider MDP models with a finite number of states and actions, it is possible to explore all the states in the MDP after a finite number of steps.

Modeling with events. If the user prefers to model a DTMDP using events, she additionally has to define a set of active events at each state i when action a is taken, $\mathcal{E}(i, a)$, and the probabilities of each event e happening at state i , $q(i, e)$. Similarly, the user must define the reachable states conditional on each event $\mathcal{S}(i, a, e)$, and define event dependent costs $c(i, a, e)$ and probabilities $p_{ij}(a, e)$. When the user asks jMDP to generate the model, the module automatically calculates the non-event-dependent versions of the model parameters as

$$c(i, a) = \sum_{e \in \mathcal{E}(i, a)} c(i, a, e) \cdot q(i, e), \quad p_{ij}(a, e) = \sum_{e \in \mathcal{E}(i, a)} p_{ij}(a, e) \cdot q(i, e), \quad \mathcal{S}(i, a) = \bigcup_{e \in \mathcal{E}(i, a)} \mathcal{S}(i, a, e).$$

Then jMDP executes Algorithm 2 to generate the state space, and the solver uses the non-event-dependent version of the parameters. In this case the user must extend the corresponding class with the *ev* suffix, e.g., DTMDPEv. A detailed example of modeling a DTMDP with events in jMDP is presented in [Riaño et al. b, Section 4], available at [jMarkov website 2016]. A similar process is followed when modeling FHMDPs with events.

Modeling with actions that depend on events. For models where actions are allowed to depend on events, the user follows the same process as above. However, in this case Algorithm 2 requires an additional modification to perform the required state expansion, because the state space of the underlying MC is made of all possible state-event combinations. To handle this jMDP uses the class StateEvent to represent state-event pairs. For each state in the initial set \mathcal{U} , jMDP generates a set of state-event pairs $\mathcal{U}'(i)$ and defines a new initial set \mathcal{U}' , respectively as

$$\mathcal{U}'(i) = \bigcup_{e \in \mathcal{E}(i)} \{i, e\}, \quad \text{and} \quad \mathcal{U}' = \bigcup_{i \in \mathcal{U}} \mathcal{U}'(i).$$

A similar modification is applied to the reachable method, so that for each state-event pair $\{i, e\}$ it returns all the reachable state-event pairs. Thus, the *MDPBuildRS* Algorithm receives \mathcal{U}' as the initial set, and uses the modified version of the reachable method to do the state space expansion.

Before solving the model, jMDP calculates $c(i, a)$ and $p_{ij}(a)$ for the new state space of state-event pairs. As a result, when the user asks for the solution, an action is assigned to each state-event pair instead of just a state. This makes the state augmentation transparent to the user, as this process is completely handled by jMDP, and avoids distorting the modeling process with this technical feature. This is handled by the jMDP module by means of a family of classes with the suffix `EvA`, for example `DTMDPEvA`.

Continuous-time models. When modeling a CTMDP the user faces the additional challenge of having to transform the problem into an equivalent DTMDP, because the standard solution algorithms for MDPs cannot be applied to CTMDPs directly. Thus, the user typically needs to model the CTMDP, find the equivalent DTMDP, solve the equivalent DTMDP, and convert the obtained solution back to the original CTMDP. Instead, the jMDP module allows the user to model the problem directly as a CTMDP. jMDP automatically determines the equivalent DTMDP using the standard Uniformization algorithm [Serfozo 1979], solves the resulting DTMDP model using an appropriate DTMDP solver and presents the solution to the user in terms of the original CTMDP problem, making the entire uniformization process seamless. This algorithm is implemented in a special class called `CT2DTConversion`. The algorithm is available for continuous-time models with or without events. To model CTMDPs the user must extend the classes `CTMDP`, `CTMDPEv` or `CTMDPEvA`.

4.2.2. Solvers. The solvers package follows a similar hierarchical structure as the modeling package. At the top we have the abstract class called `Solver`, which defines methods such as `solve` that need to be implemented by any solver. Next, we include intermediate abstract classes for families of solvers, in order to define methods used often only once, making the most of the object-oriented architecture of jMarkov. For example the class `AbstractDiscountedSolver` defines a method called `future`, which calculates the expected one-step discounted value for a state-action combination. This method is used by all the solvers for discounted problems, but not by other solvers. In the following we detail the solution algorithms implemented in jMDP, each of them corresponding to a class at the bottom of the hierarchy in the `Solvers` package.

The backward induction algorithm [Bellman 1957] is the default algorithm for FHMDPs. The class `FiniteSolver` implements it to compute and return an optimal policy. The value iteration algorithm [Puterman 1994] is implemented in the class `ValueIterationSolver`. This is set as the default solver for infinite-horizon discounted problems, and allows activating or deactivating the Gauss-Seidel modification [Hastings 1971], and the use of error bounds [Bertsekas 1995]. For discounted problems, the policy iteration algorithm [Puterman 1994] has also been implemented in the class `PolicyIterationSolver`, where the MTJ library [Heimsund 2003] is used for the matrix operations. This class also allows the user to switch to the modified policy iteration algorithm [Puterman and Shin 1978] by simply setting a parameter and specifying the number of iterations for the internal loop of the algorithm.

For the average-cost optimization criterion, the relative value iteration algorithm [Puterman 1994] is set as the default. The algorithm is implemented in the class `RelativeValueIterationSolver`. For this type of problems we also provide an implementation of policy iteration and modified policy iteration in the class `PolicyIterationAvgSolver`. Finally, for the total-cost infinite-horizon problem, the default solver is the stochastic shortest path approach to MDPs [De Alfaro 1999], which we implement in the class `StochasticShortestPathSolver`.

Additionally, jMDP supports the use of linear programming (LP) software packages to solve infinite horizon problems under either the average cost or discounted cost criteria. In this case, one defines an equivalent LP model of the problem at hand, solves its dual, and constructs an optimal deterministic policy by observing which variables take strictly positive values. A detailed explanation of our implementation can be found in [Bello and Riaño 2006]. In particular, the classes `MpsLpAverageSolver` and `MpsLpDiscountedSolver` can be used for solving average and discounted cost problems, respectively. Each of these classes builds the dual LP model of the corresponding MDP in a Mathematical Programming System (MPS) file, which can in turn be read by an LP solver to solve the problem. To facilitate this process these classes are abstract and must be extended to communicate with the specific LP solver. We provide examples using the free `QSOpt` LP solver [Applegate et al. 2003] and `XpressMP` [Dash Optimization 2005].

Finally, the extensible nature of the jMarkov framework allows the user to add new solver classes. This can include writing custom solvers for a specific class of problems, e.g. exploiting the problem structure; or creating a simple solver class to link to an existing solver the user has access to. Further details on the architecture of the module can be found in the jMDP documentation available at [jMarkov website 2016].

4.3. Related Work

To the best of our knowledge, all the tools currently available for MDPs are only *solvers*. jMDP is unique because it offers *modeling* capabilities as well as *solvers* for MDPs. More precisely, the solvers available in existing tools require the user to provide a three-dimensional array of transition probabilities and a matrix of costs as input, together with the horizon and cost criterion. Due to the large number of state-action combinations, or the even larger number of state-event-action combinations when the problem is modeled with events, the process of constructing these inputs is generally cumbersome and error prone. In order to use existing tools, the user must perform this process separately on her own. However, with jMDP this step is unnecessary as the user only defines the states, events, and actions, along with the rules that determine which events and actions are available at each state, and the corresponding transition probabilities and costs. jMDP takes care of building the state space and all other parameters of the model in the background. Furthermore, available tools do not support models with events, nor actions that depend on events. Thus, if a user builds a mathematical model using events, she must convert that model into a model without events and convert the conditional probabilities and costs into unconditional parameters in order to use the available tools. Similarly, if she wants to allow actions to depend on events, she must build the new state space of state-event pairs on her own, and convert the parameters accordingly. With jMDP these conversions are carried out in the background, so the process is seamless for the user.

Another important distinction between jMDP and existing solvers is its implementation of the Uniformization algorithm, which allows users to model and solve CTMDP problems. All the algorithms commonly used to solve CTMDP problems require finding the DTMDP analogue of the original problem, solving that problem, and then converting the solution back to the terms of the original CTMDP. In order to use any of the currently available tools to solve MDPs for a continuous-time problem, the user needs to find the DTMDP analogue *a priori* and then use the solver on that problem. With jMDP this process is seamless, to the point that the user does not need to manipulate the discrete-time problem at all, since the solution is presented in terms of the original CTMDP.

Currently available general purpose MDP solvers include MDPtoolbox [Chadès et al. 2014], available for MATLAB, R, GNU Octave, and Scilab, which offers multiple algorithms for solving DTMDP. PyMDPtoolbox [Cordwell 2013] is a Python implementation of that toolbox. Another tool for MATLAB, called MDP Toolbox [Murphy 2002] is similar to MDPtoolbox, but has fewer algorithms implemented and is no longer supported. These three solvers require the state and action spaces to be a subset of the integers, which can make parsing a solution to the original state space a hassle if such representation is not natural for a specific problem. This is not an issue in jMDP, where the user can define states and actions more generally. These tools also lack jMDP's object-oriented architecture, which allows the user to easily modify the model, which is useful, for example, to do sensitivity analyses or to calculate additional measures of performance. Furthermore, thanks to jMDP's extensibility a user could implement a new solver that calls any of the functions in these toolboxes to solve models constructed using jMDP.

Besides the aforementioned general purpose toolboxes, there exist some tools that implement a specific algorithm or focus on a specific problem. SPUD [Hoey et al. 1999] uses the SPUD algorithm, which only works for discounted DTMDPs and requires the state to be represented as a boolean vector, limiting its applicability. PRISM [Kwiatkowska et al. 2011] is a tool for probabilistic model-checking, which can check several structural properties of MDPs, DTMCs and CTMCs, but is not designed to determine optimal policies or performance measures, thus it is outside the scope of jMDP. Caylus [Teichteil-Königsbuch et al. 2010] implements several heuristic algorithms for planning problems, using aggregation-disaggregation techniques to get solutions quickly. However, its scope is limited to problems that conform to that particular structure, and it offers no guarantee of optimality. We should note here that there are several tools available for solving Partially Observable Markov Decision Processes, which in some cases can be used to solve MDPs. However, since these usually implement approximation algorithms and offer limited support for pure MDP problems, we consider them to be outside the scope of the jMDP module.

5. THE JPHASE MODULE

Phase-type (PH) distributions [Neuts 1981; Latouche and Ramaswami 1999] are a powerful tool for stochastic modeling as they are able to represent a broad range of probabilistic behaviors. Further, PH distributions can be introduced into Markovian models, which are suitable for efficient numerical analysis, replacing the more common exponential distribution. The jMarkov framework includes the jPhase module to represent PH distributions, as well as to support their inclusion in CTMC models. In addition, jPhase provides methods to obtain PH distributions from data traces and to generate PH random variates. The next sections introduce PH distributions and detail the features of jPhase.

5.1. Background

A continuous PH distribution [Neuts 1981] is defined as the time until absorption in an absorbing CTMC with m transient states. Letting A be the $m \times m$ subgenerator matrix associated with the transient states, and α their initial probability distribution, the cumulative distribution function of the time to absorption is given by $F(t) = 1 - \alpha e^{At} \mathbf{1}$, $t \geq 0$, and its moments can be obtained as $E[X^k] = k! \alpha (-A^{-1})^k \mathbf{1}$, $k \geq 1$. We refer to this distribution or the associated random variable as $\text{PH}(\alpha, A)$. Due to their Markovian description, PH distributions can be used instead of the exponential distribution in Markovian models, capturing more general behaviors [Latouche and Ramaswami 1999].

Analogously, discrete PH distributions represent the distribution of the number of steps until absorption in an absorbing Discrete Time Markov Chain (DTMC). Also, PH distributions, both continuous and discrete, enjoy being closed under a number of operations, including convolution, order statistics, and convex mixtures, among others. These closure properties can be exploited in the modeling of, for instance, manufacturing systems [Riaño 2002].

5.2. Features and Algorithms

The jPhase module offers three main sets of features: manipulation of PH distributions, including closure properties; fitting procedures to obtain PH distributions from data traces; and PH random-variate generation. These sets of features correspond to the three main packages in the module: jPhase, jPhaseFit, and jPhaseGenerator.

5.2.1. Manipulation and Closure. The central functionality of the jPhase module is the manipulation of PH variables. This entails the definition of PH variables as Java objects, and the computation of relevant metrics. jPhase thus offers methods to compute moments, such as mean or variance, as well as to evaluate the cumulative distribution function or the density probability function of a PH random variable. As the set of PH distributions includes many common special cases, such as the exponential, Erlang or hyper-exponential distributions, jPhase also provides methods to define these special cases based on their specific parameters. Furthermore, jPhase offers dense and sparse storage for the parameters of the PH distribution, vector and matrix, making use of the classes available in the MTJ library [Heimsund 2005].

Since PH variables have a number of closure properties that can be used for modeling purposes, jPhase also provides support for a broad set of these properties. For instance, the minimum or the maximum of a set of PH variables is also PH-distributed, as is the sum of two PH variables. To illustrate this, let us consider a production line where the service is made up of two stages. The first stage follows a $\text{PH}(\alpha, A)$ distribution with parameters

$$\alpha = [0.5 \ 0.3 \ 0.2] \text{ and } A = \begin{bmatrix} -4 & 2 & 1 \\ 1 & -3 & 1 \\ 2 & 1 & -5 \end{bmatrix},$$

while the second stage is an $\text{Erlang}(n, \lambda)$ distribution with parameters $\lambda = 1.5$ and $n = 2$, and the arrivals follow a Poisson process. Assuming the production line faces a traffic intensity (the ratio between the arrival and the service rates) of 0.5, we are interested in knowing the probability that the incoming jobs face a waiting time between 1 and 2 time units. The following code snippet shows how this can be obtained with jPhase.

```

1 DenseMatrix A = new DenseMatrix(new double[][] { {-4,2,1} , {1,-3,1} , {2, 1,-5} } );
2 DenseVector alpha = new DenseVector(new double[] {0.5, 0.3, 0.2});
3 DenseContPhaseVar v1 = new DenseContPhaseVar(alpha, A);
4 ContPhaseVar v2 = DenseContPhaseVar.Erlang(1.5, 2);
5 ContPhaseVar v3 = v1.sum(v2);
6 double rho = 0.5;
7 ContPhaseVar v4 = v3.waitingQ(rho);
8 double prob = v4.prob(1,2);

```

Here the first two lines define the parameters α and A , while the third line builds a dense (storage) representation of this PH variable. The fourth line defines the Erlang distribution for the second stage of service. Next we build the full representation of the service time ($v3$) by summing the variables corresponding to the two service stages just defined. We can then apply another closure property since the waiting time distribution in a first-come-first-serve M/PH/1 queue (Poisson arrivals, PH services, 1 server) is also PH. We thus call the `waitingQ` method in jPhase to obtain the PH representation of the waiting time as variable $v4$. Finally, we use the method `prob` to obtain the desired probability.

5.2.2. Fitting. To incorporate a PH distribution in an MC model a necessary step is to determine the parameters (α, A) . Consider for instance the parameterization of a queueing model representing a

server in a software application. The simplest option to model the processing times at this server is to assume they follow an exponential distribution, and set the parameter (rate) of the exponential distribution to match the measured mean processing time. However, the exponential assumption may be far from the actual processing times, for instance the real distribution may have a much shorter or longer tail than the one assumed by the exponential distribution. To incorporate this more general behavior we can instead assume PH-distributed processing times, for which we must set the parameters (α, A) . jPhase therefore offers a set of *fitting* routines, which aim to find the parameters (α, A) that best fit a set of measurements or statistical data. These routines are provided in the jPhaseFit package, which also provides a set of interfaces, abstract classes, and other methods to allow the interested user to incorporate new fitting methods. We now describe the fitting methods already provided by the jPhaseFit package, which we classify in maximum-likelihood and moment-matching techniques.

Maximum-Likelihood Algorithms. Maximum-likelihood methods employ optimization methods to find the parameters (α, A) of a PH distribution that are most-likely to generate a given data trace. All the methods of this type implemented in jPhase rely on the Expectation-Maximization (EM) algorithm [Dempster et al. 1977]. jPhaseFit implements, in class EMPhaseFit, the method proposed in [Asmussen et al. 1996], which considers the general class of continuous PH distributions as the target for the fitting method. This method is however computationally expensive as the E-step of the EM algorithm requires solving a set of $n(n+2)$ linear differential equations for a distribution of n phases. To deal with this issue, alternative methods consider specific subclasses of PH distributions as their target set. The method in [El Abdouni Khayari et al. 2003], implemented in class EMHyperExpoFit, focuses on hyper-exponential distributions, which are convex mixtures of exponential distributions and can be represented by a PH distribution with a diagonal matrix A . A similar, though more general, approach was proposed in [Thümmler et al. 2005], which targets hyper-Erlang distributions. These are convex mixtures of Erlang distributions that include the hyper-exponential as a special case, but also allow for hypo-exponential behaviors, such as the Erlang distribution. This method is implemented by the EMHyperErlangFit class.

Moment-Matching Algorithms. Moment-matching methods aim to find the parameters (α, A) such that the PH distribution has a set of predefined moments (mean, variance, asymmetry, etc). Such methods are very useful in analyzing the impact that a given moment, for instance the variance, of a certain quantity has on the measures of performance. In the production line example, one can be interested in determining the impact that the variability of the processing times, measured by their variance, has on the total time to produce an item, including queuing times. jPhaseFit offers four methods of this type, three for continuous PH distributions, and one for discrete ones. The method in [Telek and Heindl 2002] matches the first three non-centered moments with a two-phase PH distribution, as long as these are representable. jPhaseFit implements both the continuous and discrete variants of this method in the classes MomentsACPH2Fit and MomentsADPH2Fit. jPhaseFit also implements, in class MomentsECCompleteFit, the method proposed in [Osogami and Harchol 2006], which determines the minimum number of phases needed to represent any set of three non-centered moments attainable by a distribution with non-negative support. This feature is shared by the method proposed in [Bobbio et al. 2005], which is implemented in class MomentsACPHFit.

5.2.3. Random-Variate Generation. While the main focus of jMarkov is on the numerical solution of MC models, we also recognize that an alternative tool for modeling and analysis is to use stochastic simulation. In fact, in many applications simulation is more appropriate because of the complex relationships between different stochastic variables. While PH distributions have gained popularity, they are still not mainstream and are therefore not supported by most simulation tools. jPhase therefore includes support for PH random-variate generation in the jPhaseGenerator package. The classes in this package offer implementations of the methods proposed in [Neuts and Pagano 1981] for both continuous and discrete PH distributions. Since jMarkov is distributed as a Java library, its random-variate generators can be incorporated in simulation models developed in this language, for instance using the SSJ library [LEcuyer and Buist 2005].

5.3. Related Work

All tools available for PH distributions concentrate on *fitting*, and most are standalone implementations accompanying the papers where the corresponding fitting method was introduced. For instance, EMPht [Olsson 1998] implements the method in [Asmussen et al. 1996] to fit general continuous PH distributions, while PHFit implements the fitting methods in [Horváth and Telek 2002] for both discrete and continuous PH distributions. Also, G-Fit implements the method proposed in [Thümmler et al. 2006] for hyper-Erlang distributions, a sub-class of PH distributions, and HyperStar provides an implementation of the methods in [Reinecke et al. 2012] with a graphical interface to fine-tune the

results. Different from the tools above, jPhase provides a range of methods for PH fitting, which allows the user to select the one that best fits his needs, or to implement new methods.

A more comprehensive tool is Butools [Butools 2016], which also offers a range of methods for fitting PH distributions, as well as methods to compute moments and probability functions. As mentioned in Section 3.3, Butools also offers solvers for QBD processes and other structured MC models. However, jPhase is unique in implementing a wide set of closure properties useful for modeling, as well as in providing random-variate generation methods. More importantly, jPhase is part of the jMarkov modeling framework, which allows the user to exploit PH variables, built from data and/or by means of closure properties, to construct finite MC and QBD models from first principles. The next section provides an illustrative example.

6. NUMERICAL EXPERIMENTS

To validate the algorithms implemented in jMarkov we have performed exhaustive tests on jMarkov. In [jMarkov website 2016] we currently provide over 20 tests, in the form of jUnit [JUnit 2016] test cases, where we compare the results of many models against results obtained with alternative tools. These test cases cover the main module, as well as the jQBD, jMDP, and jPhase modules.

Table II. Execution times (sec) with jMarkov and SMC Solver for the priority queue model

| C | m | jMarkov | | | SMCSolver | | |
|------|------|---------|-------|-------|-----------|-------|-------|
| | | ρ | | | ρ | | |
| | | 0.1 | 0.5 | 0.9 | 0.1 | 0.5 | 0.9 |
| 100 | 202 | 0.38 | 0.36 | 0.34 | 0.06 | 0.07 | 0.08 |
| 200 | 402 | 1.76 | 2.26 | 2.49 | 0.44 | 0.37 | 0.45 |
| 500 | 1002 | 31.12 | 31.10 | 29.87 | 13.99 | 10.83 | 9.82 |
| 1000 | 2002 | 252.5 | 292.4 | 297.7 | 63.70 | 144.9 | 134.6 |
| 2000 | 4002 | 2151 | 2298 | 2586 | 435.5 | 659.3 | 997.7 |

To illustrate the execution times that can be expected with jMarkov we make use of the priority queue example introduced in the accompanying user manual [Riaño et al. a, Section 6], available at [jMarkov website 2016]. We set the arrival rates to achieve three levels of server utilization ρ , namely 0.1, 0.5, and 0.9. The utilization is the fraction of time that the server is busy and influences the execution time of the logarithmic-reduction algorithm used to solve the QBD model. Also, we set the service time distributions to have 2 phases, and we consider values for the buffer capacity C between 100 and 2000. Table II reports the size m of the A_i matrices that define the QBD in (1). Table II also compares the execution times, in seconds, obtained with jMarkov against those obtained with SMCSolver [Bini et al. 2009], which executes in MATLAB. These times were obtained on a MAC with a 2.9 GHz Intel Core i7 with 2 cores, 8 GB of memory, and running OS X 10.11.4. First we note that jMarkov is able to solve large systems, with block sizes up to 4000. jMarkov is however about 2-5 times slower than SMCSolver. This comes at no surprise since SMCSolver relies on the very efficient matrix manipulation in MATLAB. In spite of this, jMarkov is well-suited to consider middle-sized problems, requiring less than a second to solve problems with block size 200, and just 30 seconds for problems with block size 1000.

As we have highlighted, the key benefits of jMarkov lie in its modeling capabilities, but we have seen above that its solvers are also well-suited for middle-sized problems. In addition, the flexibility of jMarkov, and its implementation in Java, offer a number of options to solve larger problems, or to exploit the reduced execution times offered by tools like SMCSolver on MATLAB. It is possible for instance to generate the MC/QBD model in jMarkov, save the associated matrices in files, which are then loaded in a different tool to solve the model. Alternatively, in the following code snippet we show how to directly import the jMarkov routines from MATLAB to exploit the solvers in SMCSolver. The first step is to include the jMarkov library in the static Java classpath of MATLAB. This allows us, as shown in the first 4 lines of the code snippet, to import the jMarkov routines, as well as the MTJ library for some matrix manipulations. In particular, the third line imports the model `PriorityQueueMPPHPPreempt` we defined in the previous section. Lines 6-14 define the model parameters in a similar manner as in the previous section, exploiting the jPhase routines to fit PH distributions. Line 16 creates the model object and line 17 generates the model parameters. The next step, in lines 19-21, is to extract the matrices that define the QBD from the jMarkov model object. Finally, we use these matrices to call the routines in SMCSolver to solve the model, as in lines 23-24. This illustrates that the modeling capabilities of jMarkov can be exploited to build complex models, and the user can easily employ alternative solvers if this is required or preferred.

```

1 import jphase.fit.EMHyperErlangFit;
2 import jphase.fit.MomentsACPHFit;
3 import examples.jmarkov.PriorityQueueMPHPHPreempt;
4 import no.uib.cipr.matrix.Matrices;
5
6 lambda_hi = 0.2; lambda_low = 0.2;
7 bufferCapacity = 10;
8
9 data = csvread('data/W2.txt');
10 fitter_hi = EMHyperErlangFit(data);
11 servTime_hi = fitter_hi.fit(4);
12
13 fitter_low = MomentsACPHFit(2, 6, 25);
14 servTime_low = fitter_low.fit();
15
16 model = PriorityQueueMPHPHPreempt(lambda_hi, lambda_low, servTime_hi, servTime_low, bufferCapacity);
17 model.generate();
18
19 As = model.getAMatrices(); Bs = model.getBMatrices();
20 A0 = Matrices.getArray(As(1)); A1 = Matrices.getArray(As(2)); A2 = Matrices.getArray(As(3));
21 B00 = Matrices.getArray(Bs(1)); B01 = Matrices.getArray(Bs(2)); B10 = Matrices.getArray(Bs(3));
22
23 [G,R] = QBD_LR(A2,A1,A0);
24 pi = QBD_pi(B10,B00, R, 'Boundary', [B01; A1 + R*A2]);

```

7. APPLICATIONS AND EXTENSIONS

To conclude the paper we would like to mention some applications of the jMarkov framework to specific problems, which exploit the modeling capabilities of jMarkov to build complex MC, QBD and MDP models from basic rules. We also highlight some extensions that rely on the ease-of-extensibility built in to jMarkov. [Bello and Riaño 2006] discusses how jMarkov can be effectively connected to commercial software packages to solve MDPs as Linear Programming problems. [Medina et al. 2007] uses jMDP to build a model that reduces the cost of issuing a mortgage-backed security by changing the structure of the security issued. The model was implemented in a real-world instance, using the original data of a Colombian securitizing firm, which allowed the authors to determine the different levels of improvement attained by the model. [Riaño et al. 2006] uses jMarkov to model a bucket-brigades system [Bartholdi and Eisenstein 1996] with stochastic processing times and to compare it to the basic deterministic model in the original paper. [Silva et al. 2009] uses jMarkov to model congestion in manual order-picking-systems and compare layouts with narrow and wide aisles. [Ucrós Ospino 2012] uses jMarkov to consider PH distributions in the modeling of a semiconductor manufacturing system as a queueing network. Most recently, [Gomez et al. 2013] uses jPhase to fit PH distributions to travel and service time data, as part of a stochastic vehicle routing problem. These results highlight the ability of jMarkov to support researchers in the development and solution of complex models.

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